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On the axisymmetric loading of an annular crack by a disk inclusion

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Abstract. This paper examines the axisymmetric elastostatic problem related to the loading of an annular crack by a rigid disk-shaped inclusion subjected to a central force. The integral equations associated with the resulting mixed-boundary-value problem are solved numerically to determine the load-displacement result for the rigid inclusion and the Mode II stress-intensity factors at the boundaries of the annular crack. The results presented are applicable to a wide range of Poisson's ratios ranging from zero to one half.

Key words: annular crack, crack-inclusion interaction, inclusion problem, three-part boundary-value problem, triple integral equations

1. Introduction

The annular crack represents an idealized form of a *flattened toroidal* crack, which has been investigated quite extensively in connection with problems in fracture mechanics. The annular crack also represents a general case from which solutions to both penny-shaped cracks and external circular cracks can be recovered as special cases (see *e.g.*, Sneddon and Lowengrub [1, Chapter 3]; Kassir and Sih [2, Chapter 1]; Cherepanov [3, Appendix A]). The axisymmetric annular-crack problem examined by Grinchenko and Ulitko [4] is based on an approximate solution of the governing equations. Smetanin [5] examined the problem of the axisymmetric axial loading of a flat toroidal crack and used an asymptotic expansion method to obtain estimates for the stress-intensity factors at the crack tip. Moss and Kobayashi [6] have utilized the approach proposed by Mossakovski and Rybka [7] to develop iterative approximate stress-intensity factors at the crack boundaries. The solution procedure for the axisymmetric problem of an annular crack employed by Shibuya *et al.* [8] reduces the problem to the solution of an infinite system of algebraic equations. Choi and Shield [9] have presented an elegant analysis of the types of problems where the annular crack is subjected to axisymmetric deformations induced by torsionless axisymmetric and torsional loads. These authors use Betti's reciprocal theorem to derive the integral equations governing the plane annular crack. This study also provides an estimate of the accuracy of the solutions developed by Smetanin [5], and Moss and Kobayashi [6]. An external crack problem for a cylinder was also investigated by Nied and Erdogan [10] where numerical values for the stress intensity factor are given.

The analysis of the annular crack problem presented by Selvadurai and Singh [11] reduces the three-part mixed-boundary-value problem to the solution of a pair of coupled integral equations of the Fredholm type, which are solved using power-series representations of the unknown functions in terms of a non-dimensional parameter corresponding to the radii ratio.

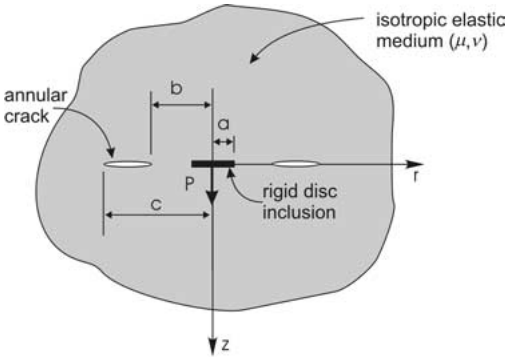

Figure 1. Loading of annular crack by an embedded rigid-disk inclusion.

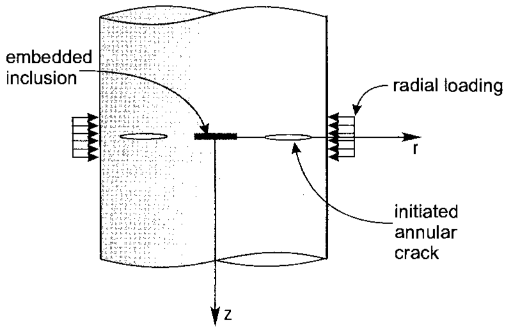

Figure 2. Annular crack development during radial loading of an embedded spheriodal rigid conclusion.

In the case where the loading of the crack is symmetric about the plane of the annular crack the analysis of the problem is formally similar to that of the contact problem involving smooth indentation of the halfspace by an annular rigid punch. The three-part boundary-value problems encountered in applied mathematics can be solved by appeal to a variety of approximate and numerical techniques. The methods outlined in [12–16] [17, Chapter 6] and [18] essentially reduce the three-part boundary-value problem to the solution of a Fredholm integral equation. Clements and Ang [19] have examined the annular-crack problem by employing the procedures proposed by Clements and Love [20] for the solution of the potential problem referred to an annulus. The paper by Clements and Ang [19] also represents a comparison of the estimates for the stress-intensity factors at the boundaries of the uniformly loaded annular crack, which are available in the literature.

In this paper we examine the axisymmetric elastostatic problem dealing with an annular crack where the intact central region contains a disk-shaped rigid inclusion (Figure 1). The rigid-disk inclusion can be visualized as a flattened oblate spheroidal rigid inclusion which is bonded to the surrounding elastic medium (Figure 2). In multi-phase components, where the interface bonding is enhanced with a fortifier, the inclusion-matrix interface can possess strength and fracture-toughness characteristics, which are greater than those of the matrix, and any development of defects usually occurs in the matrix region. An annular crack at the plane of symmetry can result from radially symmetric loadings applied in the plane of the disk inclusion (Figure 2). The inclusion region contained within the annular crack is subjected to an axial displacement, which induces a state of asymmetry in the deformation about the plane $z = 0$. This induces Mode II-type of stress-intensity factors at the boundaries of the annular crack. The mode of deformation, although highly idealized in its representation, can be induced by a force field associated with centrifugal effects. The problem of the axisymmetric interaction between the annular crack and the central rigid-disk inclusion is reduced to a mixed-boundary-value problem for a halfspace region. The problem is effectively reduced to the solution of a system of coupled Fredholm integral equations of the second-kind. These equations are solved via a numerical technique to determine the force-displacement relationship for the rigid disk and the purely Mode II stress intensity factors at the extremities of the annular crack.

2. Governing equations

The solution of the mixed-boundary-value problem associated with the axisymmetric loading of the annular crack can be approached via several formulations in terms of special stress and displacement functions applicable to the classical theory of elasticity, including the Boussinesq-Neuber-Papkovich functions ([21, Chapter 5], [22], [23, Chapter 1]). We consider here the method proposed by Love [24, Chapter 8] that is particularly suited for the class of axisymmetric deformations associated with an isotropic elastic medium. In Love's approach, the solution to the axisymmetric problem in classical elasticity can be represented in terms of a single strain function that satisfies the *biharmonic equation*. The relevant expressions for the displacement and stress components in terms of Love's strain function $\varphi(r, z)$ referred to the cylindrical polar coordinate system (r, θ, z). In the absence of body forces $\varphi(r, z)$ satisfies

$$\nabla^2\nabla^2\varphi(r, z) = 0\,, \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is the axisymmetric form of Laplace's operator referred to the cylindrical polar coordinate system. The relevant expressions for the displacement and stress components take the forms

$$2\mu u_r(r, z) = -\frac{\partial^2\varphi}{\partial r\partial z}, \quad 2\mu u_z(r, z) = 2(1-\nu)\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2} \tag{3}$$

and

$$\sigma_{zz}(r, z) = \frac{\partial}{\partial z}\left[(2-\nu)\,\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2}\right], \quad \sigma_{rz}(r, z) = \frac{\partial}{\partial r}\left[(1-\nu)\,\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2}\right] \tag{4}$$

respectively, where ν is Poisson's ratio and μ is the linear elastic shear modulus.

In view of the asymmetry of the deformation induced during movement of the in-plane rigid-disk inclusion contained within the intact region $r \in (0, b)$, the inclusion-annular-crack problem related to an elastic infinite space can be formulated as a mixed-boundary-value problem referred to a halfspace region, $z \geq 0$. We therefore seek solutions of (1) which also satisfy the regularity conditions, which require the displacement and stress fields to reduce to zero as $r, z \longrightarrow \infty$. Employing a Hankel-transform development of the governing partial differential equation (1) ([25, Chapter 10], [26, Chapter 8]) we obtain the following solution

$$\varphi(r, z) = \int_0^\infty \left[\frac{A\,(\xi)}{\xi^2} + \frac{z}{\xi}B(\xi)\right] \mathrm{e}^{-\xi z} J_0\,(\xi r)\,\mathrm{d}\xi \tag{5}$$

where $A(\xi)$ and $B(\xi)$ are arbitrary functions. The expressions for the relevant displacement and stress components obtained from (5) and evaluated at $z = 0$ take the forms

$$\begin{aligned} 2\mu u_r(r, 0) &= \int_0^\infty [-A\,(\xi) + B(\xi)]\, J_1\,(\xi r)\,\mathrm{d}\xi \\ 2\mu u_z(r, 0) &= -\int_0^\infty [A\,(\xi) + 2\,(1-2\nu)\,B(\xi)]\, J_0\,(\xi r)\,\mathrm{d}\xi \end{aligned} \tag{6}$$

and

$$\begin{aligned} \sigma_{zz}(r,0) &= \int_0^\infty \xi\,[A\,(\xi) + (1-2\nu)\,B(\xi)]\,J_0\,(\xi r)\,\mathrm{d}\xi \\ \sigma_{rz}(r,0) &= \int_0^\infty \xi\,[A\,(\xi) - 2\nu B(\xi)]\,J_1\,(\xi r)\,\mathrm{d}\xi \end{aligned} \tag{7}$$

3. The annular crack-disk inclusion interaction problem

We consider the axisymmetric problem of an open annular crack of external radius c and internal radius b where the region $r \in (0, b)$ is reinforced by a rigid-disk inclusion of radius a (Figure 2) in complete bonded contact with the surrounding elastic medium of infinite extent. The rigid-disk inclusion is subjected to an axial displacement Δ_0 in the z-direction. The annular crack is assumed to remain in an open position during the movement of the disk. The axial displacement of the rigid-disk inclusion induces a state of asymmetry in the displacement and stress fields about the plane $z = 0$. Hence we can formulate the annular crack-inclusion interaction problem as a multi-part mixed-boundary-value problem referred to a halfspace region ($z \geq 0$), where the plane $z = 0$ is subjected to the following displacement and stress boundary conditions:

$$u_z\,(r,0) = \Delta_0 \quad ; \quad 0 \leq r \leq a, \tag{8}$$

$$u_r\,(r,0) = 0 \quad ; \quad 0 \leq r \leq b, \tag{9}$$

$$u_r\,(r,0) = 0 \quad ; \quad c \leq r \leq \infty, \tag{10}$$

$$\sigma_{zz}\,(r,0) = 0 \quad ; \quad a < r < \infty, \tag{11}$$

$$\sigma_{rz}\,(r,0) = 0 \quad ; \quad b < r < c. \tag{12}$$

Using the results given by (6) and (7), we can reduce the multi-part boundary-value problem defined by (8) to (12) to the following system of integral equations in terms of two auxiliary functions $M(\xi)$ and $N(\xi)$ as follows:

$$\int_0^\infty M(\xi)\,J_0\,(\xi r)\,\mathrm{d}\xi = -2\mu\Delta_0 \quad ; \quad 0 \leq r \leq a, \tag{13}$$

$$\int_0^\infty N(\xi)\,J_1\,(\xi r)\,\mathrm{d}\xi = \frac{(1-2\nu)}{(3-4\nu)}\int_0^\infty M(\xi)\,J_1\,(\xi r)\,\mathrm{d}\xi \quad ; \quad 0 \leq r \leq b, \quad ; \quad c \leq r \leq \infty, \tag{14}$$

$$\int_0^\infty \xi M(\xi)\,J_0\,(\xi r)\,\mathrm{d}\xi = -\,(1-2\nu)\int_0^\infty \xi N(\xi)\,J_0\,(\xi r)\,\mathrm{d}\xi \quad ; \quad a < r < \infty, \tag{15}$$

$$\int_0^\infty \xi N(\xi)\,J_1\,(\xi r)\,\mathrm{d}\xi = 0 \quad ; \quad b < r < c, \tag{16}$$

where

$$A(\xi) = \frac{1}{(1-\nu)}\,[\nu M(\xi) + (1-2\nu)\,N(\xi)]\,, \quad B(\xi) = \frac{1}{2\,(1-\nu)}\,[M(\xi) - N(\xi)] \tag{17}$$

When the integral representations

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{rJ_0(\xi r)}{\left[t^2-r^2\right]^{1/2}}\mathrm{d}r = \cos(\xi t) \tag{18}$$

and

$$\int_t^\infty \frac{rJ_0(\xi r)}{\left[r^2-t^2\right]^{1/2}}\mathrm{d}r = \frac{\cos(\xi t)}{\xi} \tag{19}$$

are used, the Equations (13) and (15) yield the following:

$$M(\xi) = \frac{2}{\pi}\left[-\frac{2\mu\Delta_0 \sin(\xi a)}{\xi} + \int_a^\infty F(u)\cos(\xi u)\,\mathrm{d}u\right], \tag{20}$$

where

$$F(t) = -(1-2\nu)\int_0^\infty N(\xi)\cos(\xi t)\,\mathrm{d}\xi \qquad ; \qquad a < t < \infty. \tag{21}$$

Considering the Equations (14) and (16), we assume that $N(\xi)$ admits a representation

$$\int_0^\infty \xi N(\xi)\, J_1(\xi r)\,\mathrm{d}\xi = \begin{cases} f_1(r) & ; \quad o < r < b, \\ f_2(r) & ; \quad c < r < \infty. \end{cases} \tag{22}$$

Making use of Hankel transforms, we obtain from (17)

$$N(\xi) = \int_0^b u f_1(u)\, J_1(\xi u)\,\mathrm{d}u + \int_c^\infty u f_2(u)\, J_1(\xi u)\,\mathrm{d}u \tag{23}$$

Substituting (23) in (14), we obtain

$$\int_0^b u f_1(u)\, L(u,r)\,\mathrm{d}u + \int_c^\infty u f_2(u)\, L(u,r)\,\mathrm{d}u = g(r) \qquad ; \quad 0 < r < b, \qquad ; \quad c < r < \infty, \tag{24}$$

where

$$g(r) = \frac{(1-2\nu)}{(3-4\nu)}\int_0^\infty M(\xi)\, J_1(\xi r)\,\mathrm{d}\xi \tag{25a}$$

$$L(u,r) = \int_0^\infty J_1(\xi u)\, J_1(\xi r)\,\mathrm{d}\xi \tag{25b}$$

Using the results given by Cooke [14], we can write (26) in the form

$$L(u,r) = \frac{2}{\pi u r}\int_0^{\min(u,r)} \frac{s^2\mathrm{d}s}{\left[\left(u^2-s^2\right)\left(r^2-s^2\right)\right]^{1/2}} \tag{26}$$

$$= \frac{2ur}{\pi}\int_{\max(u,r)}^\infty \frac{\mathrm{d}s}{s^2\left[\left(s^2-r^2\right)\left(s^2-u^2\right)\right]^{1/2}}, \tag{27}$$

where

$$\begin{aligned}\int_{a_1}^{b_1} \mathrm{d}u \int_0^{\min(u,r)} \mathrm{d}s &= \int_{a_1}^{r} \mathrm{d}s \int_s^{b_1} \mathrm{d}u + \int_0^{a_1} \mathrm{d}s \int_{a_1}^{b_1} \mathrm{d}u,\\ \int_{a_1}^{b_1} \mathrm{d}u \int_{\max(u,r)}^{\infty} \mathrm{d}s &= \int_{r}^{b_1} \mathrm{d}s \int_{a_1}^{s} \mathrm{d}u + \int_{b_1}^{\infty} \mathrm{d}s \int_{a_1}^{b_1} \mathrm{d}u.\end{aligned} \tag{28}$$

Using (27) and (28), we can write the Equation (24), applicable to $0 < r < b$, in the form

$$\begin{aligned}\int_0^r \frac{s^2 \mathrm{d}s}{\left(r^2 - s^2\right)^{1/2}} \int_s^b \frac{f_1(u)\,\mathrm{d}u}{\left(u^2 - s^2\right)^{1/2}} + r^2 \int_c^{\infty} \frac{\mathrm{d}s}{s^2\left(s^2 - r^2\right)^{1/2}} \int_c^s \frac{u^2 f_2(u)\,\mathrm{d}u}{\left(s^2 - u^2\right)^{1/2}}\\ = \frac{\pi r}{2} g(r) \qquad ; \qquad 0 < r < b.\end{aligned} \tag{29}$$

We now introduce functions $F_1(s)$ and $F_2(s)$ such that

$$F_1(s) = s^2 \int_s^b \frac{f_1(u)\,\mathrm{d}u}{\left(u^2 - s^2\right)^{1/2}}, \tag{30}$$

$$F_2(s) = \int_c^s \frac{u^2 f_2(u)\,\mathrm{d}u}{\left(s^2 - u^2\right)^{1/2}} \tag{31}$$

and rewrite (29) as

$$\int_0^r \frac{F_1(s)\,\mathrm{d}s}{\left(r^2 - s^2\right)^{1/2}} = -r^2 \int_c^{\infty} \frac{F_2(s)\,\mathrm{d}s}{s^2\left(s^2 - r^2\right)^{1/2}} + \frac{\pi r}{2} g(r) \qquad ; \qquad 0 < r < b. \tag{32}$$

Since the integral Equation (32) is of the Abel-type, its solution can be written as

$$\begin{aligned}F_1(s) = &-\frac{2}{\pi} \int_c^{\infty} \left[\frac{-s^2}{u\left(s^2 - u^2\right)} + \frac{s}{2u^2} \log\left|\frac{s+u}{s-u}\right| \right] F_2(u)\,\mathrm{d}u\\ &+ \frac{\mathrm{d}}{\mathrm{d}s} \int_0^s \frac{r^2 g(r)\,dr}{\left(s^2 - r^2\right)^{1/2}} \qquad ; \qquad 0 < s < b.\end{aligned} \tag{33}$$

Similarly, the Equation (24), applicable to $c < r < \infty$, gives an Abel integral equation of the form

$$\int_r^{\infty} \frac{F_2(s)\,\mathrm{d}s}{s^2\left(s^2 - r^2\right)^{1/2}} = -\frac{1}{r^2} \int_0^b \frac{F_1(s)\,\mathrm{d}s}{\left(r^2 - s^2\right)^{1/2}} + \frac{\pi}{2r} g(r) \qquad ; \qquad c < r < \infty \tag{34}$$

which has the solution

$$\begin{aligned}F_2(s) = &\frac{1}{s^2} \frac{\mathrm{d}}{\mathrm{d}s} \int_s^{\infty} \frac{g(r)\,\mathrm{d}r}{\left(r^2 - s^2\right)^{1/2}}\\ &- \frac{2}{\pi} \int_0^b \left[\frac{s}{\left(s^2 - u^2\right)} + \frac{1}{2u} \log\left|\frac{s+u}{s-u}\right| \right] F_1(u)\,\mathrm{d}u \qquad ; \qquad c < s < \infty.\end{aligned} \tag{35}$$

Using integral representations similar to (18) and (19), (25), we have

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_0^s \frac{r^2 g(r)\,\mathrm{d}r}{\left(s^2 - r^2\right)^{1/2}} = \frac{(1 - 2\nu)}{(3 - 4\nu)} s \int_0^{\infty} M(\xi) \sin(\xi s)\, d\xi \qquad ; \qquad 0 < s < b, \tag{36}$$

$$s^2 \frac{d}{ds} \int_s^\infty \frac{g(r)\,dr}{\left(r^2 - s^2\right)^{1/2}} = \frac{(1-2\nu)}{(3-4\nu)} \int_0^\infty \frac{1}{\xi} \left[\xi s \cos(\xi s) - \sin(\xi s)\right] M(\xi)\, d\xi ;\ c < s < \infty. \quad (37)$$

Substituting the value of $M(\xi)$ from (20) in (36) and (37), using a repeated application of the solution to integral equations of the Abel type and performing some lengthy algebraic manipulations, we obtain the following system of coupled Fredholm-type integral equations for the unknown functions $A_1(s)$ and $A_2(s)$

$$\begin{aligned} &A_1(s) + \frac{2(1-2\nu)^2}{\pi^2(3-4\nu)} \int_0^b \frac{s A_1(u)}{u\left(u^2 - s^2\right)} \left\{ u \log\left|\frac{a-s}{a+s}\right| - s \log\left|\frac{a-u}{a+u}\right| \right\} du \\ &\quad + \frac{2}{\pi} \int_c^\infty \left\{ -\frac{s^2}{u\left(s^2-u^2\right)} + \frac{s}{2u^2} \log\left|\frac{s+u}{s-u}\right| \right\} A_2(u)\, du \\ &\quad - \frac{2(1-2\nu)^2}{\pi(3-4\nu)} \int_c^\infty \left\{ \frac{s}{2u^2} \log\left|\frac{(u-s)(a+s)}{(u+s)(a-s)}\right| - \frac{s^2}{u\left(u^2-s^2\right)} \right\} A_2(u)\, du \\ &\quad = -\frac{2s}{\pi} \frac{(1-2\nu)}{(3-4\nu)} \log\left|\frac{s+a}{s-a}\right| \quad ; \quad 0 < s < b, \end{aligned} \quad (38)$$

$$\begin{aligned} &A_2(s) \left[1 + \frac{(1-2\nu)^2}{(3-4\nu)} \right] + \frac{2}{\pi} \int_0^b \left\{ \frac{s}{\left(s^2-u^2\right)} + \frac{1}{2u} \log\left|\frac{s+u}{s-u}\right| \right\} A_1(u)\, du \\ &\quad + \frac{2}{\pi} \frac{(1-2\nu)^2}{(3-4\nu)} \int_0^b \frac{s A_1(u)}{\left(s^2-u^2\right)} du - \frac{(1-2\nu)^2}{(3-4\nu)} a \int_c^\infty \frac{A_2(u)}{u^2} du \\ &\quad - \frac{1}{\pi} \frac{(1-2\nu)^2}{(3-4\nu)} \int_0^b \frac{A_1(u)}{u} \log\left|\frac{(s-u)(a+u)}{(s+u)(a-u)}\right| du \\ &\quad = -2a \frac{(1-2\nu)}{(3-4\nu)} \quad ; \quad c < s < \infty, \end{aligned} \quad (39)$$

where $A_1(s)$ and $A_2(s)$ are normalized expressions for $F_1(s)$ and $F_2(s)$, respectively, given by

$$A_1(s) = \frac{F_1(s)}{\Delta_0 \mu} \quad ; \quad A_2(s) = \frac{F_2(s)}{\Delta_0 \mu}. \quad (40)$$

The annular crack-disk inclusion interaction problem is thus reduced to the solution of the pair of coupled Fredholm integral equations of the second kind for the functions $A_1(s)$ and $A_2(s)$ and defined through the intervals $0 < s < b$ and $c < s < \infty$.

Results of some importance to engineering applications concerns the evaluation of the load-displacement relationship for the disk inclusion and the shearing-mode stress-intensity factors at the boundaries of the annular crack.

The axial stress in the inclusion region is given by

$$\sigma_{zz}(r,0) = \frac{1}{2(1-\nu) r} \frac{\partial}{\partial r} \left\{ r \left[\int_0^\infty M(\xi) J_1(\xi r)\, d\xi + (1-2\nu) \int_0^\infty N(\xi) J_1(\xi r)\, d\xi \right] \right\} ; \quad 0 < r < a. \quad (41)$$

The equilibrium equation for the disk inclusion gives

$$P = -2\int_0^{2\pi}\int_0^a r\sigma_{zz}(r,0)\,\mathrm{d}r\mathrm{d}\theta. \tag{42}$$

Using the results for $M(\xi)$ and $N(\xi)$ in terms of $A_1(s)$ and $A_2(s)$, we can simplify the result (42) to the form

$$\frac{P}{\mu\Delta a} = \frac{8}{(1-\nu)} - \frac{4}{a(1-\nu)}\int_a^\infty F(u)\left\{1-\frac{u}{\left(u^2-a^2\right)^{1/2}}\right\}\mathrm{d}u - \frac{4(1-2\nu)}{a(1-\nu)}\left[\int_0^b \frac{A_1(s)}{\left(a^2-s^2\right)^{1/2}}\mathrm{d}s\right.$$
$$\left. + \int_c^\infty \frac{1}{s^2}\left\{s-\left(s^2-a^2\right)^{1/2}\right\}A_2(s)\,\mathrm{d}s - \int_c^\infty\left\{1-\frac{s}{\left(s^2-a^2\right)^{1/2}}\right\}\frac{A_2(s)}{s}\mathrm{d}s\right], \tag{43}$$

where

$$F(u) = -\frac{2(1-2\nu)}{\pi}\int_c^\infty \frac{A_2(\zeta)}{\zeta}\mathrm{d}\zeta\int_0^\infty \cos(u\xi)\left\{\frac{\sin(\xi\zeta)}{\xi\zeta}-\cos(\xi\zeta)\right\}\mathrm{d}\xi$$
$$+\frac{2(1-2\nu)}{\pi}\int_0^b \frac{A_1(\eta)}{\left(u^2-\eta^2\right)}\mathrm{d}\eta \quad ; \quad a < u < \infty. \tag{44}$$

Since the deformation is asymmetric about $z = 0$, the only non-zero stress component at the tips of the annular crack is due to σ_{rz}, such that

$$\sigma_{rz}(r,0) = \frac{2rF_1(b)}{\pi b^2\left(b^2-r^2\right)^{1/2}} - \frac{2r}{\pi}\int_r^b \frac{1}{\left(s^2-r^2\right)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{F_1(s)}{s^2}\right)\mathrm{d}s; \quad 0 < r < b \tag{45}$$

and

$$\sigma_{rz}(r,0) = \frac{2F_2(c)}{\pi r\left(r^2-c^2\right)^{1/2}} + \frac{2}{\pi r}\int_c^r \frac{1}{\left(r^2-s^2\right)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}s}\left(F_2(s)\right)\mathrm{d}s; \; c < r < \infty. \tag{46}$$

The crack-shearing mode-stress intensity factors at the boundaries of the annular crack are defined by

$$K_{II}^b = \lim_{r\to b^-}[2(b-r)]^{1/2}[\sigma_{rz}(r,0) \; ; \quad 0 < r < b], \tag{47}$$

$$K_{II}^c = \lim_{r\to c^+}[2(r-c)]^{1/2}[\sigma_{rz}(r,0) \; ; \quad c < r < \infty]. \tag{48}$$

Using the expressions for $\sigma_{rz}(r,0)$ given by (45) and (46) in (47) and (48), respectively, we obtain

$$K_{II}^b = \frac{2}{\pi}\frac{F_1(b)}{b^{3/2}} = \frac{\Delta_0 E A_1(b)}{\pi b^{3/2}(1-\nu)}, \tag{49}$$

$$K_{II}^b = \frac{2}{\pi}\frac{F_2(c)}{c^{3/2}} = \frac{\Delta_0 E A_2(c)}{\pi c^{3/2}(1-\nu)}, \tag{50}$$

where E is Young's modulus of the elastic medium.

4. Numerical solution of the integral equations

The coupled Fredholm integral equations of the second kind (38) and (39) governing the annular crack-disk inclusion interaction problem are not amenable to solution in an exact form. In this section we outline a procedure that can be used to solve these integral equations in a numerical fashion (see [27, Chapter 5] and [28, Chapter 10]) to evaluate the load-displacement relationship for the embedded rigid disk and to establish the influence of the loading on the Mode II stress-intensity factors at the boundaries of the annular crack. We can rewrite (38) and (39) in the general forms

$$R_1 A_1^*(s) + \int_0^b A_1^*(u) K_{11}(u,s)\mathrm{d}u + \int_c^\infty A_2^*(u) K_{12}(u,s)\mathrm{d}u = f_1(s);\ 0 \le s \le b, \tag{51}$$

$$R_2 A_2^*(s) + \int_0^b A_1^*(u) K_{21}(u,s)\mathrm{d}u + \int_c^\infty A_2^*(u) K_{22}(u,s)\mathrm{d}u = f_2(s);\ c \le s \le \infty, \tag{52}$$

where we have assumed that $A_1(u)$ and $A_2(u)$ admit representations of the form

$$[A_1(u);\ A_2(u)] = -\frac{2u(1-2\nu)}{\pi(3-4\nu)}[A_1^*(u);\ A_2^*(u)] \tag{53}$$

and the kernel functions K_{11}, K_{12}, etc. are defined by

$$\begin{aligned} K_{11}(u,s) &= \left\{\frac{C_1}{(u^2-s^2)}\{u \log\left|\frac{a-s}{a+s}\right| - s\log\left|\frac{a-u}{a+u}\right|\right\},\ K_{22}(u,s) = C_2\frac{a}{su} \\ K_{12}(u,s) &= K_{21}(u,s) = C_3\left\{-\frac{s}{\left(s^2-u^2\right)} + \frac{1}{2u}\log\left|\frac{s+u}{s-u}\right|\right\} + C_4\frac{1}{2u}\log\left|\frac{a+s}{a-s}\right|. \end{aligned} \tag{54}$$

In (51–54) the constants R_1, R_2, C_1, C_2, etc. are given by

$$\begin{aligned} R_1 &= 1 &&;\ R_2 = \frac{4(1-\nu)^2}{(3-4\nu)} \\ C_1 &= \frac{2(1-2\nu)^2}{\pi^2(3-4\nu)} &&;\ C_2 = -\frac{(1-2\nu)^2}{(3-4\nu)} \\ C_3 &= \frac{8(1-\nu)^2}{\pi(3-4\nu)} &&;\ C_4 = -\frac{2}{\pi}\frac{(1-2\nu)}{(3-4\nu)} \end{aligned} \tag{55}$$

and the functions $f_1(s)$ and $f_2(s)$ are defined by

$$f_1(s) = \log\left|\frac{s+a}{s-a}\right|;\ f_2(s) = \frac{\pi a}{s} \tag{56}$$

For the numerical evaluation of the load-displacement relationship (43) we require the function $F(u)$ defined by (44), which can be further simplified to

$$\begin{aligned} F(u) = {} & \frac{2(1-2\nu)}{\pi}\int_0^b \frac{A_1(\eta)}{(u^2-\eta^2)}\mathrm{d}\eta - (1-2\nu)\int_c^\infty \frac{A_2(\zeta)}{\zeta^2}H(\zeta-u)\mathrm{d}\zeta \\ & +(1-2\nu)\frac{A_2(u)}{u}H(u-c), \end{aligned} \tag{57}$$

where $H(\cdot)$ is the Heaviside step function. Substituting the expressions (53) in (57) and using the result in (43), we obtain the following expression for the normalized axial stiffness of the rigid disk inclusion as

$$\frac{P(3-4\nu)}{32\mu\Delta_0 a(1-\nu)} = \frac{(3-4\nu)}{r(1-\nu)^2} + \frac{(1-2\nu)^2}{4\pi(1-\nu)^2}\left\{\int_0^b \frac{A_1^*}{\pi a}\log\left|\frac{a+u}{a-u}\right|\mathrm{d}u + \int_c^\infty \frac{A_2^*(u)}{u}\mathrm{d}u\right\}. \quad (58)$$

The stress-intensity factors at the locations $r = b$ and $r = c$ are given by

$$K_{II}^b = -\frac{4(1-2\nu)}{\pi^2(3-4\nu)}\frac{\mu\Delta_0}{\sqrt{b}}A_1^*(b), \quad (59)$$

$$K_{II}^c = -\frac{4(1-2\nu)}{\pi^2(3-4\nu)}\frac{\mu\Delta_0}{\sqrt{c}}A_2^*(c). \quad (60)$$

For the numerical solution of (51) and (52) we discretize the intervals $[0, b]$ and $[c, \infty]$ into N_1 and N_2 segments, respectively. In the interval $[0, b]$, N_1 segments are of equal length, whereas in the interval $[c, \infty]$ the segment sizes are increased proportionally, so that the upper limit of infinity can be approximated by a large number. The discretized forms of (51) and (52) can be written as

$$[R^*\delta_{ij} + A_{ij}]\{X_j\} = \{f_i\} \quad (61)$$

with $i, j = 1, 2, \ldots, N$, where $N = N_1 + N_2$. The coefficients of A_{ij} are given by
(i) $K_{11}(u, s)$ when $i \geq 1$; $j \leq N_1$,
(ii) $K_{12}(u, s)$ when $1 \leq i \leq N_1$; $N_1 < j \leq N$,
(iii) $K_{21}(u, s)$ when $N_1 < i < N$; $1 \leq j \leq N_1$
and
(iv) $K_{22}(u, s)$ when $N_1 < i$; $j \leq N$.
The vector on the right-hand side of (61) is given by
(v) f_1 when $1 \leq i \leq N_1$,
and
(vi) f_2 when $N_1 < i \leq N$.
The unknowns are $A_1^*(u)$ for $1 \leq i \leq N_1$ and $A_2^*(u)$ for $N_1 < i \leq N$. The coefficient R^* in (61) is such that $R^* = 1$ when $1 \leq i \leq N_1$ and $R^* = R_2$ when $N_1 < i \leq N$. For the diagonal terms A_{ij} with $1 \leq i \leq N_1$, limiting values of $K_{11}(u, s)$ are considered, *i.e.*,

$$u \overset{\text{Lim}}{\rightarrow} s\ K_{11}(u, s) = \frac{C_1}{2s}\left[\frac{2as}{(a^2-s^2)} + \log\left|\frac{a-2}{a+s}\right|\right]. \quad (62)$$

Upon solving the matrix equation (61) for the vector of unknowns $\{X_j\}$, we use the discretized values to determine the load-displacement relationship (58) and the normalized stress intensity factors

$$\overline{K}_{11}^b = \frac{4\pi^2 b^{3/2}(1-\nu)K_{11}^b}{(1-2\nu)P_0}, \quad \overline{K}_{11}^c = \frac{8\pi c^{3/2}(1-\nu)K_{11}^c}{(1-2\nu)P_0}, \quad (63)$$

where

$$P_0 = \frac{32\mu\Delta_0 a(1-\nu)}{(3-4\nu)} \quad (64)$$

represents the load-displacement relationship for a disk inclusion embedded in an infinite space region void of any cracks (see, *e.g.* [29–31]). The result (64) can also be recovered as a special case of the present problem where we set $b \to c$. In this case the problem can be simplified at the outset, with

$$M(\xi) = -\frac{4\Delta_0\mu\sin(\xi a)}{\pi\xi} \tag{65}$$

and

$$\sigma_{zz}(r,0) = -\frac{8\Delta_0\mu(1-\nu)}{\pi(3-4\nu)}\frac{1}{(a^2-r^2)^{1/2}}. \tag{66}$$

The substitution of (66) in (42) gives the result (64) with the appropriate interpretation of P_0.

5. Numerical results

The Figures 3a–d illustrate the variation in the normalized axial load P/P_0 required to induce an axial displacement of Δ_0 in the disk inclusion. Figure 3a also presents the results for the case $b/c \in (0{\cdot}01, 0{\cdot}08)$ and $a/b \in (0,1)$. In the case where $b/c = 0{\cdot}01$, and $a/b \to 1{\cdot}0$ we can interpret the solution as a limiting case where the inclusion is embedded in bonded contact within two identical halfspace regions. For this limiting case the exact closed-form solution can be deduced from the result for the load-displacement relationship for a rigid punch which is bonded to the surface of a halfspace region, which was obtained by Mossakovski [32] and Ufliand [33] (see also [23, Chapter 10]). The analysis of the bonded-punch problem yields the following results for the axial stiffness of an inclusion in bonded contact between two halfspace regions

$$\lim_{\substack{(b/a \to 1)\\(c/a\to\infty)}} \left\{\frac{P(3-4\nu)}{32\mu\Delta_0 a(1-\nu)}\right\} = \frac{(3-4\nu)\log(3-4\nu)}{4(1-\nu)(1-2\nu)}. \tag{67}$$

In the limit as $\nu \to 1/2$, the normalized stiffness reduces to unity. In this case the solution *also* corresponds to the result for the axial stiffness of a rigid-disk inclusion which is embedded in smooth contact between two halfspace regions. (Implicit in such a model is the requirement for a pre-compression of the inclusion to enable the development of tensile contact stresses without separation during axial displacement of the inclusion.) Also, the result (67) for the case as $(b/a) \to 1$ and $(c/a) \to \infty$ is derived by a Hilbert-problem approach for the analysis of the adhesive contact between the rigid circular punch and an elastic halfspace region. Such an analysis accounts for the oscillatory form of the stress singularity at the boundary of the bonded rigid punch. In the case of the annular crack-inclusion interaction problem, the stress singularities both at the boundaries of the crack and at the boundary of the rigid-disk inclusion are regular and of the $1/\sqrt{r}$-type. In the limit as $(b/a) \to 1$, this order of the singularity is maintained without account for any oscillatory singularities. Selvadurai [34] has examined the influence of the form of the stress singularities (oscillatory vs. regular) on the axial stiffness of the bonded rigid circular punch. In the case where the regular stress singularity is incorporated in the bonded contact problem, the analysis can be reduced to the solution of a single Fredholm integral equation of the second kind, which can be solved numerically. It is shown in [34] that, in the limiting case when $\nu \to 0$, the influence of the nature of the singularity will result in a difference of approximately 0·05% in the calculation

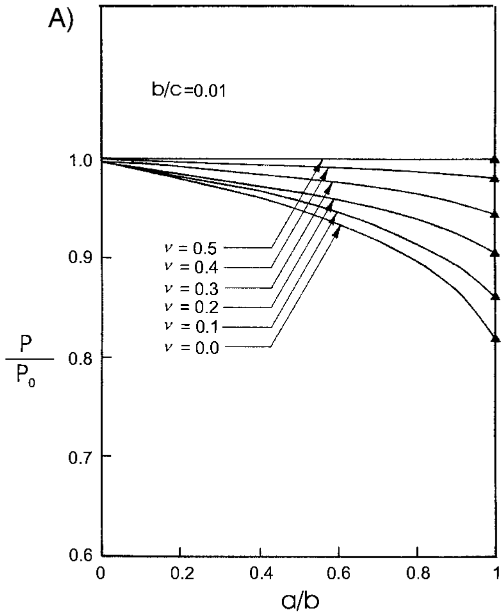

Figure 3a. Influence of the annular crack on the displacement of the bonded disk.

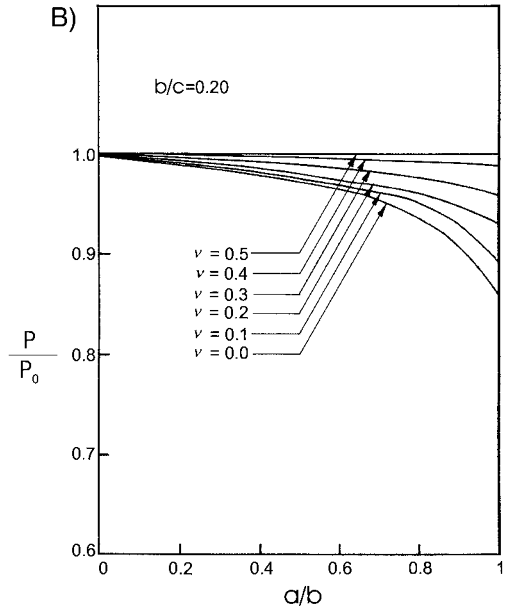

Figure 3b. Influence of the annular crack on the displacement of the bonded disk.

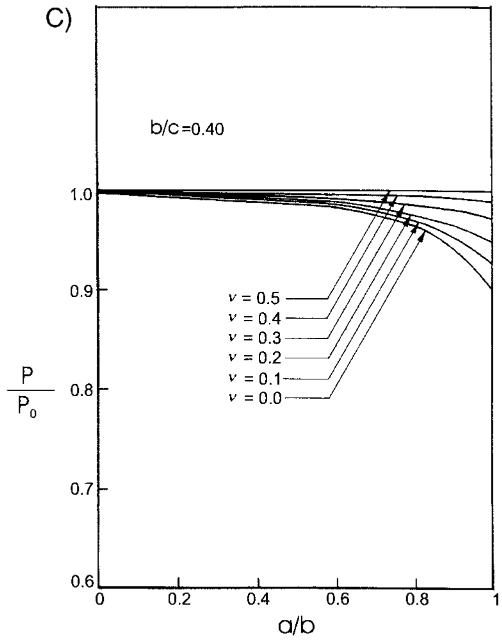

Figure 3c. Influence of the annular on the displacement of the bonded disk.

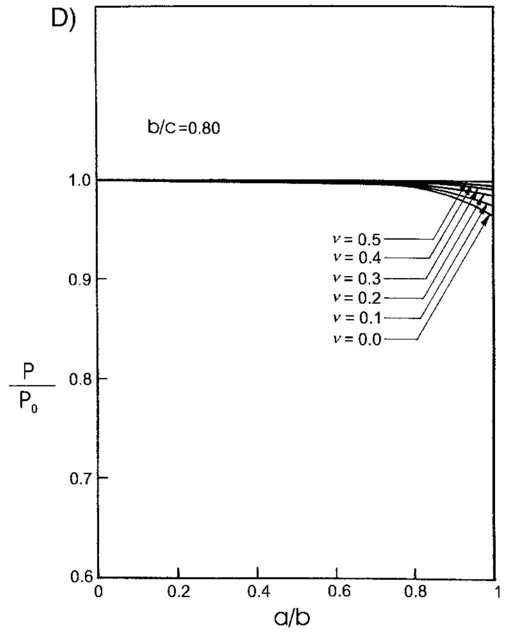

Figure 3d. Influence of the annular crack on the displacement of the bonded disk.

of the axial stiffness. In the limit when $\nu \to 1/2$, the oscillatory form of the stress singularity at the boundary of the bonded region reverts to the regular $1/\sqrt{r}$-type and the computed axial stiffness estimates are identical. These same observations apply to the case where, in the limit as $(b/a) \to 1$ and $(c/a) \to \infty$, the solution reduces to that of the axial displacement of a disk inclusion in bonded contact with two halfspace regions.

Figures 4a–d, illustrate the variations in the normalized Mode II stress-intensity factor $\overline{K}_{11}^{b}$ at the boundary $r = b$ of the annular crack for various values of ν, a/b and b/c. The results are computed for only a value of a/b approaching unity. As previously indicated, at the limit $a/b = 1$, the traction-free region of the elastic medium terminates at the boundary of the inclusion; the stresses at the boundary $a/b = 1$ will have an oscillatory form. While the omission of the oscillatory form of the stress singularity has only a marginal effect in the calculation of the axial stiffness, it is expected to be important in the calculation of stress-intensity factors at the boundary of the adhesively connected region. In this case, the stress intensity factors can be calculated separately by using the oscillatory forms of the stress singularity associated with the bonded flat punch problem. Figures 5a–d illustrate the corresponding results for the normalized Mode II stress-intensity factor at the boundary $r = c$ of the annular crack. Since the non-dimensional forms for $\overline{K}_{11}^{b}$ and $\overline{K}_{11}^{c}$ are normalized with respect to b and c, respectively, a direct comparison of the results is not warranted. We can rewrite the ratio of the stress intensity factors K_{11}^{b} and K_{11}^{c} in the form

$$\frac{K_{11}^{c}}{K_{11}^{b}} = \frac{1}{2}\left(\frac{b}{c}\right)^{2/3}\frac{\overline{K}_{11}^{c}}{\overline{K}_{11}^{b}}, \tag{68}$$

where the values of $\overline{K}_{11}^{b}$ and $\overline{K}_{11}^{c}$ are given in Figures 4a–5d. It can be shown that for the range of values of $(b/c) \in (0{\cdot}01, 0{\cdot}8)$, $\nu \in (0, 0{\cdot}5)$ and $(a/b) \in (0, 0{\cdot}99)$,

$$K_{11}^{c} < K_{11}^{b}, \tag{69}$$

indicating that during quasi-static loading of the elastic medium by the embedded rigid inclusion, a circumscribing annular crack is most likely to extend, in the Mode II, at the circular boundary closest to the inclusion. Selvadurai and Singh [35] examined the problem of the axisymmetric loading of an annular crack by a doublet of concentrated forces spaced at a distance $2h$. In this case, the location of Mode I crack extension will depend upon the ratio h/b, b/c and ν.

6. Concluding remarks

The class of problems that deals with the interaction of cracks and inclusions is of some interest to the study of the mechanics of composite materials reinforced particulate solids. Cracks can be generated in such reinforced materials due to a variety of mechanical and environmental effects. The generalized analysis of the crack-inclusion interaction problem is not amenable to a rigorous analytical treatment and solutions can be obtained through some numerical procedure based on boundary-integral equation or finite-element techniques. For simplified geometries dealing with planar cracks and planar inclusions, some progress can be made to develop analytical results. Even with such simplifications, the problems that can be analyzed are few in number. This paper examined the axisymmetric interaction between an annular crack and a disk-shaped rigid inclusion that is located in the central intact region of

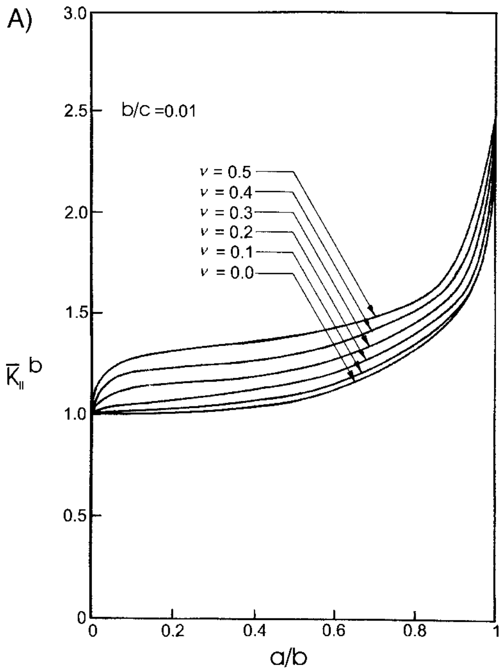

Figure 4a. Normalized Mode II stress intensity factor at the crack tip, $r = b$.

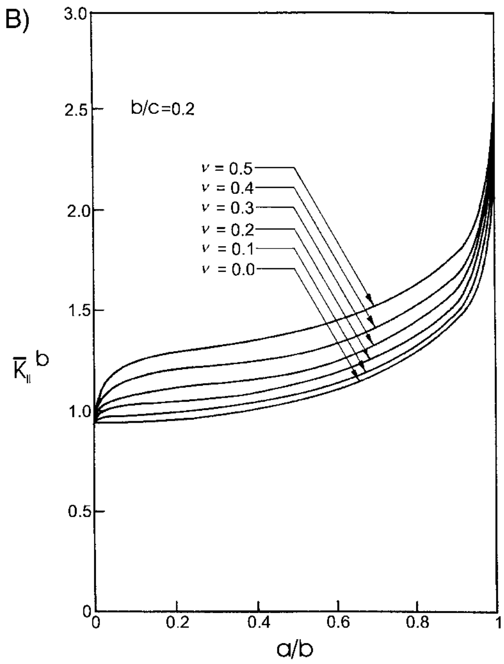

Figure 4b. Normalized Mode II stress intensity factor at the crack tip, $r = b$.

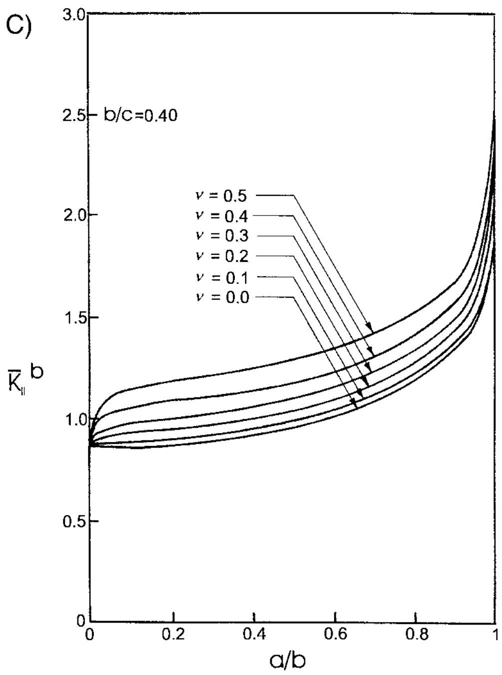

Figure 4c. Normalized Mode II stress intensity factor at the crack tip, $r = b$.

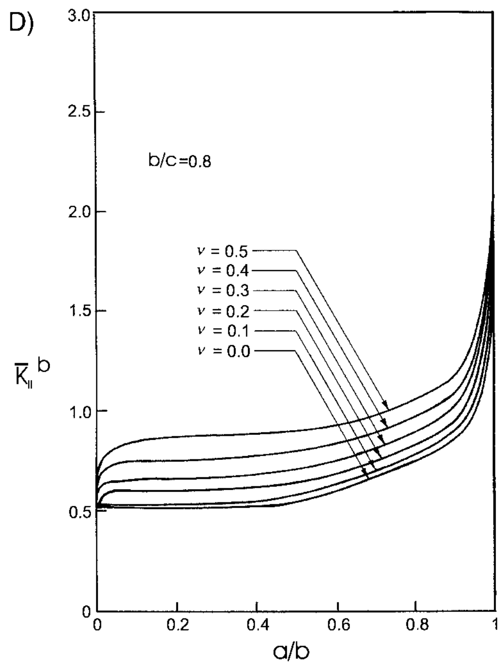

Figure 4d. Normalized Mode II stress intensity factor at the crack tip, $r = b$.

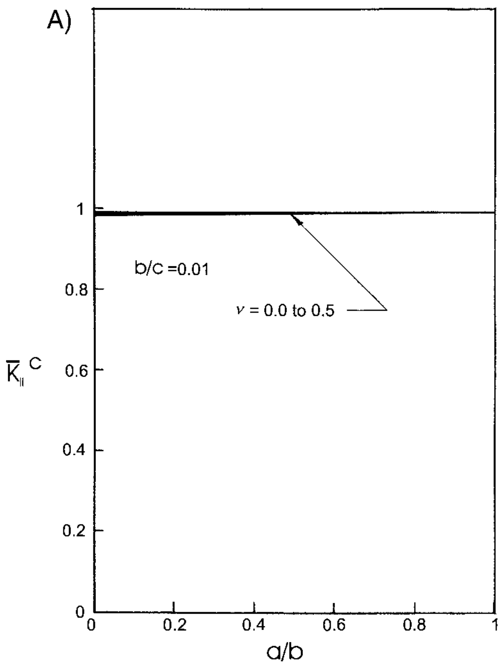

Figure 5a. Normalized Mode II stress intensity factor at the crack tip, $r = c$.

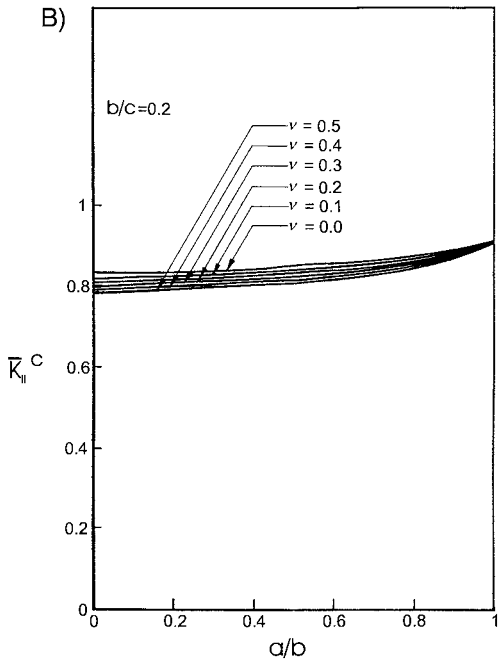

Figure 5b. Normalized Mode II stress intensity factor at the crack tip, $r = c$.

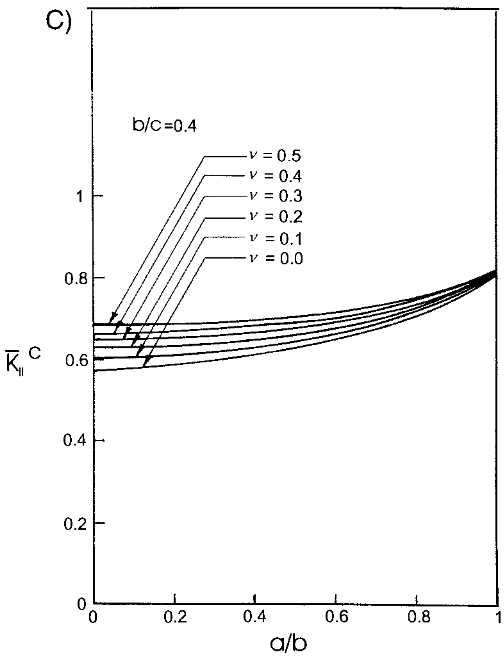

Figure 5c. Normalized Mode II stress intensity factor at the crack tip, $r = c$.

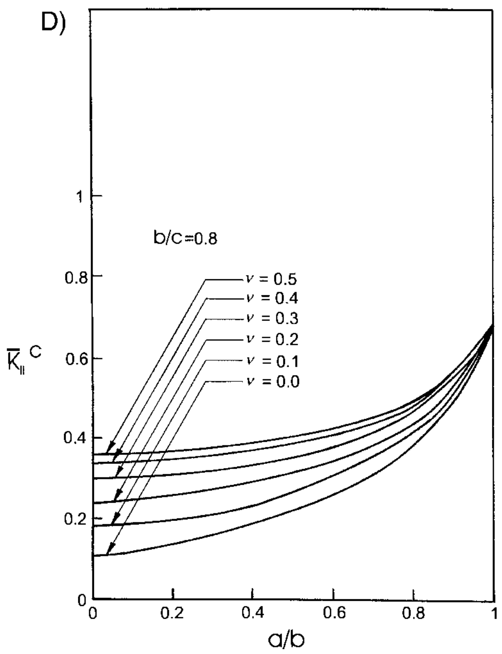

Figure 5d. Normalized Mode II stress intensity factor at the crack tip, $r = c$.

the annular crack. It is shown that, when the annular crack is loaded by the axial translation of the disk inclusion, the resulting elastostatic problem can be effectively reduced to the solution of two coupled Fredholm integral equations of the second kind. The solution of these equations can be achieved in a number of ways and in this paper the equations were reduced to a system of algebraic equations where a quadrature technique was used to evaluate the associated integrals. In particular, it was shown that the axial stiffness of the disk inclusion and the Mode II stress-intensity factors at the tip of the crack can be evaluated to a sufficient degree of accuracy that enables comparison with other results for appropriate limiting cases available in the literature. It was shown that the larger of the Mode II stress-intensity factors occurs at the inner boundary of the annular crack, indicating that during axisymmetric Mode II dominated loading, the annular crack will extend to the boundary of the rigid-disk inclusion. As a likely scenario, if the bond between the inclusion and the elastic medium is sound, further crack extension will now take place at the outer boundary of the annular region. If on the other hand, the bond between the disk inclusion and the elastic medium is weak, the crack will extend through detachment at one of its faces. In this case the problem reduces to that of the loading of a penny-shaped crack through the axial loading of a rigid-disk inclusion that is bonded to one of its faces. Solutions to this category of crack-disk inclusion interaction problem are available in the literature [36, 37].

References

1. I.N. Sneddon and M. Lowengrub, *Crack Problems in the Classical Theory of Elasticity*. New York: John Wiley (1969) 221 pp.
2. M.K. Kassir and G.C. Sih, *Mechanics of Fracture Vol. 2 Three Dimensional Crack Problems*. Leiden, The Netherlands: Noordhoff Int. Publ. Co. (1974) 452 pp.
3. G.P. Cherepanov, *Mechanics of Brittle Fracture* (Translation edited by R. de Witt and W.C. Cooley). New York: McGraw-Hill (1979) 939 pp.
4. V.T. Grinchenko and A.F. Ulitko, Tension of an elastic space weekend by a ring-shaped crack. *Prikl. Mekh.* 1 (1965) 61–64.
5. B.I. Smetanin, Problem of extension of an elastic space containing a plane annular slit. *J. Appl. Math. Mech.* 32 (1968) 458–462.
6. L.W. Moss and A.S. Kobayashi, Approximate analysis of axisymmetric problems in fracture mechanics with application to a flat toroidal crack. *Int. J. Fract. Mech.* 7 (1971) 89–99.
7. V.I. Mossakovski and M.T. Rybka, Generalization of the Griffith-Sneddon criterion for the case of a non-homogeneous body. *Prikl. Math. Mekh.* 28 (1964) 1061–1069.
8. T. Shibuya, T. Koizumi and I. Nakahara, An elastic contact problem for a half-space indented by a flat annular rigid stamp. *Int. J. Engng. Sci.* 2 (1974) 759–771.
9. I. Choi and R.T. Shield, A note on a flat toroidal crack in an isotropic body. *Int. J. Solids Struct.* 18 (1982) 479–486.
10. H.F. Nied and F. Erdogan, The elasticity problem for a thick-walled cylindar containing a circumferential crack. *Int. J. Fract.* 22 (1978) 277–301.
11. A.P.S. Selvadurai and B.M. Singh, The annular crack problem for an isotropic elastic solid. *Q. J. Mech. Appl. Math.* 38 (1985) 233–243.
12. B. Noble, On certain dual integral equations. *J. Math. Phys.* 37 (1958) 128–136.
13. C.J. Tranter, Some triple integral equations. *Proc. Glasgow Math Assoc.* 4 (1960) 200–203.
14. J.C. Cooke, Some further triple integral equation solutions. *Proc. Edin. Math. Soc.* 13 (1963) 317–323.
15. W.E. Williams, Integral equation formulation of some three-part boundary value problems. *Proc. Edin. Math. Soc.* 13 (1963) 317–323.
16. W.D. Collins, On the solution of some axisymmetric boundary value problems by means of integral equations. IV. Potential problems for circular discs. *Proc. Edin. Math. Soc.* 13 (1963) 235–246.

17. I.N. Sneddon, *Mixed Boundary Value Problems in Potential Theory*. Amsterdam: North-Holland (1966) 283 pp.
18. D.L. Jain and R.P. Kanwal, Three-part boundary value problem in potential and generalized axially symmetric potential theories. *J. d' analyse math.* 25 (1972) 107–158.
19. D.L. Clements and W.T. Ang, Stress intensity factors for the circular annular crack. *Int. J. Engng. Sci.* 26 (1988) 325–358.
20. D.L. Clements and E.R. Love, Potential problems involving an annulus. *Proc. Camb. Phil. Soc.* 76 (1974) 313–325.
21. A.E. Green and W. Zerna, *Theoretical Elasticity*. Oxford: Clarendon Press (1968) 457 pp.
22. M.E. Gurtin, The linear theory of elasticity. In: C. Truesdell (ed.), *Handbuch der Physik, Mechanics of Solids II*. (1972) 1–295.
23. G.M.L. Gladwell, *Contact Problems in the Classical Theory of Elasticity*. Alphen aan den Rijn, The Netherlands: Sijthoff and Noordhoff (1980) 716 pp.
24. A.E.H. Love, *A Treatise on the Classical Theory of Elasticity*. Cambridge: Cambridge University Press (1927) 643 pp.
25. I.N. Sneddon, *Fourier Transforms*. New York: McGraw-Hill (1951) 542 pp.
26. A.P.S. Selvadurai, *Partial Differential Equations in Mechanics Vol. 2. The Biharmonic Equation, Poisson's Equation*. Springer-Verlag, Berlin (2000) 698 pp.
27. C.T.H. Baker, *The Numerical Treatment of Integral Equations*. Oxford: Clarendon Press (1977) 1034 pp.
28. L.M. Delves and J.L. Mohamed, *Computational Methods for Integral Equations*. Cambridge: Cambridge University Press (1985) 376 pp.
29. W.D. Collins, Some axially symmetric stress distributions in elastic solids containing penny-shaped cracks. I. Cracks in an infinite solid, *Proc. R. Soc. London A* 203 (1962) 359–386.
30. R.P. Kanwal and D.L. Sharma, Singularity methods in elastostatics. *J. Elasticity* 6 (1976) 405–418.
31. A.P.S. Selvadurai, The load-deflection characteristics of a deep rigid anchor in an elastic medium. *Geotechnique* 26 (1976) 603–612.
32. V.I. Mossakovski, The fundamental mixed boundary problem of the theory of elasticity for a half-space with a circular line separating the boundary conditions. *Prikl. Math. Mekh.* 18 (1954) 187–196.
33. Ia.S. Ufliand, The contact problem of the theory of elasticity for a die, circular in its plane, in the presence of adhesion. *Prikl. Math. Mekh.* 20 (1956) 578–587.
34. A.P.S. Selvadurai, Influence of a boundary fracture on the elastic stiffness of a deeply embedded anchor plate. *Int. J. Num. Analytical Meth. Geomech.* 13 (1989) 64–77.
35. A.P.S. Selvadurai and B.M. Singh, Body force loading of an annular adhesive contact region. *Int. J. Fracture* 29 (1985) 109–204.
36. A.P.S. Selvadurai and B.M. Singh, Axial displacement of a disk inclusion embedded in a penny-shaped crack. *J. Appl. Math. Phys. (ZAMP)* 37 (1985) 64–77.
37. A.P.S. Selvadurai, On the problem of a detached anchor plate embedded in a crack. *Int. J. Solids Structs.* 31 (1994) 1279–1290.